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ON PERFECT MATROID DESIGNS

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ABSTRACT. We give a short survey of known results and some new extremal properties of PMD.

Let us consider a matroid M of rank r on a finite set V (the definitions of matroid and related concepts be found, for example, in the recent book [13]). Denote F^i any flat (closed set) of rank i, i.e., any i-flat. For any i $(0 \le i \le r)$ denote M^i the set of all F^i. Thus the hyperplane family of M will be M^{r-1}. The set $\bigcup_{i=0}^{r} M^i$ is a lattice and it contains the intersection of any two flats.

A <u>perfect matroid design</u> (<u>PMD</u> for short) is a matroid such that for any integer i, $0 \le i \le r$, the cardinality $|F^i|$ depends only on i. Let us denote $|F^i| = \alpha_i$. PMD's were introduced and deeply studied by Murty, Young and Edmonds in [9], [10]. They were called also "MYED's" (by initials of authors), "matroid designs", "geometric designs" by different authors.

But we will reserve (following [13]) the term <u>matroid design</u> for a matroid whose hyperplane family M^{r-1} forms the set of blocks of a BIBD (balanced incomplete block design). PMD is a special case of matroid design. Information and bibliography on matroid designs is given in Ch.12.3 of [13]. (There are also considered base designs, i.e., matroids whose bases (maximal independent sets) are the blocks of BIBD).

PMD's form a proper subset of the set of all equicardinal matroids, i.e. matroids with the same cardinalities of hyperplanes. Equicardinal matroids were first studied in [5] and later in [11].

A PMD of rank 3 is a BIBD with $\lambda = 1$; a PMD of higher rank is a very special BIBD. In the basic paper [9] was proved the following

Proposition 1.

a) The set M^{r-1} of all hyperplanes of a PMD is the set of blocks of a BIBD on the set of 1-flats (points) of this PMD;

b) The set of independent sets of given cardinal is the set of blocks of a BIBD;

c) The set of circuits of given cardinal is the (possibly vacuous) set of blocks of a BIBD.

Moreover in [9], [10] it was shown that the corresponding sets are sets of blocks of some t-design, $t \geq 2$. Perhaps one day (when we will know more about the existence of PMD's) this will be a good tool for constructing new t-designs.

Proposition 1 and the following list of all known PMD's show that we have to consider PMD's more as part of design theory than of matroid theory.

Below we consider only <u>simple</u> matroid designs, i.e., $\alpha_0 = 0$ ($F^0 = \emptyset$), $\alpha_1 = 1$ (without loss of generality for our purposes).

Let F^i, F^ℓ some flats of PMD, $0 \le i < \ell \le r$. In [9] it was proved that $F^\ell \backslash F^i$ is partitioned by sets of the form $F^{i+1} \backslash F^i$ (where F^{i+1} is an $(i+1)$-flat such that $F^i \subseteq F^{i+1} \subseteq F^\ell$). Let us denote $t(i,j,\ell)$ the number of j-flats F^j such that $F^i \subseteq F^j \subseteq F^\ell$.

<u>Proposition 2 ([9])</u>. The number $t(i,j,\ell)$ is independent of the choice of F^i and F^ℓ.

Actually, $t(0,1,i) = \alpha_i$; $t(0,i,r) = \prod_{j=1}^{i-1} \frac{v-\alpha_j}{\alpha_i-\alpha_j}$; $t(i,i+1,\ell) = \frac{\alpha_\ell - \alpha_i}{\alpha_{i+1}-\alpha_i}$ and, in general, $t(i,j,\ell) = \frac{t(i,f,\ell)\cdot t(f,j,\ell)}{t(i,f,j)}$ for $0 \le i \le f \le j \le \ell \le r$.
It is interesting that the number $t(0,i,r)$ considered as a polynomial $t(x)$ of argument $x=r$ and having degree $r-1$ satisfies the condition $t(\alpha_i) = 1$, $t(\alpha_0) = t(\alpha_1) = \cdots = t(\alpha_{i-1}) = 0$ (it is Lagrange's formula of interpolation of functions by polynomials). From Proposition 2 it follows that a matroid associated with the <u>interval</u> $[F,G]$ of a lattice of all flats of a given PMD is also a PMD (here F,G are any flats of a given PMD with $F \subseteq G$). In particular, any <u>truncation</u> of PMD is a PMD. <u>Submatroid of</u> PMD <u>on</u> some of its flats F^i (or, <u>reduction of</u> PMD to F^i) is also a PMD.

All known PMD's are those given below and their truncations.

1) A Steiner system $S(t,k,v)$ (i.e. t-design $S_1(t,k,v)$) is a PMD of rank $t+1$. Actually M^i is the set of all i-subsets of V for $0 \le i \le t-1$ and the hyperplane family $M^t = M^{r-1}$ is the set of all blocks of $S(t,k,v)$.

2) We will consider separately trivial Steiner systems $S(k,k,v)$, because in this case any M^i $(0 \le i \le r-1=k)$ is the set of all i-subsets of V. We will call $S(k,k,v)$ trivioid; sometimes people call it "uniform matroid" or "truncation of boolean algebra".

3) An affine geometry $AG(r,q)$ is a PMD of rank r. Actually the i-flats F^i are subspaces of dimension i.

4) A projective geometry $PG(r,q)$ is a PMD of rank r. Also the i-flats are subspaces of dimension i.

5) We will call Hall-Young matroids the PMD's of rank 4 with $\alpha_2 = 3$, $\alpha_3 = 9$, $|V| = \alpha_4 = 3^a$ (for any integer $a \ge 4$). An example for $a = 4$ was given in [4] and generalized in [12]. It was constructed as erection of special affine triple systems.

We denote $L = \{\alpha_i : 0 \le i \le r-2\}$, $k = \alpha_{r-1}$, $v(=|V|) = \alpha_r$. In these notations we can see that

1) $L = \{0,1,\ldots,r-2\}$, $k = r-1$ for the trivioid $S(k,k,v)$ of rank r;

2) $L = \{0,1,q,q^2,\ldots,q^{r-2}\}$, $k = q^{r-1}$, $v = q^r$ for $AG(r,q)$;

3) $L = \{0,1,\frac{q^2-1}{q-1},\ldots,\frac{q^{r-1}-1}{q-1}\}$, $k = \frac{q^r-1}{q-1}$, $v = \frac{q^{r+1}-1}{q-1}$ for $PG(r,q)$;

4) $L = \{0,1,\ldots,t-1\}$ for a Steiner system $S(t,k,v)$;

5) $L = \{0,1,3\}$, $k = 9$, $v = 3^a$ $(a \geq 4)$ for Hall-Young matroids.

It was shown in [10] that any PMD with parameters L,k,v as given in 1) - 4) is necessarily a trivioid, affine space, projective space or Steiner system corresp. It was shown in [12] that there is a unique Hall-Young matroid with $v = 3^4 = 81$. In terms of the parameters L,k,v, i.e., $\alpha_0,\alpha_1,\ldots,\alpha_2$ necessary conditions of existence of PMD, given in [10], take the following form

Proposition 3. For any PMD we have

1) $\prod_{f=i+1}^{j} \frac{\alpha_\ell - \alpha_{f-1}}{\alpha_j - \alpha_{f-1}}$ is integer for $0 \leq i \leq j \leq \ell \leq r$;

2) $(\alpha_i - \alpha_{i-1}) \mid (\alpha_{i+1} - \alpha_i)$ for $2 \leq i \leq r-1$;

3) $(\alpha_i - \alpha_{i-1})^2 \leq (\alpha_{i+1} - \alpha_i)(\alpha_i - \alpha_{i-1})$ for $r \leq i \leq r-1$.

These conditions are not sufficient for the existence of PMD with given sequence $\alpha_0,\alpha_1,\ldots,\alpha_2$. An example is provided (R. Wilson) by the sequence 0,1,3,7,43. (Actually he showed that any PMD with $\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_2 \geq 3$, $(\alpha_3 - \alpha_2) = (\alpha_2 - 1)^2$ has the property $\alpha_r(\alpha_2 - 2) = (\alpha_2 - 1)^h - 1$ for some integer h).

For the first special case of PMD rank 4 we know only the following necessary condition: if there exists an infinity of numbers v such that there exists a PMD with $L = (0,1,\ell)$, $|F^{r-1}| = k$, $|F^r| = v$ then there exists $S(2,\ell,k)$. For $\ell=2$ such infinity exists; it will be an infinity of $S(3,k,v)$ given by the theorem of R. Wilson. For first next case $\ell=3$ we have the existence of Steiner triple system as necessary condition. Such infinity exists for $L = \{0,1,3\}$, $k=7$ or 9 (PG or AG). First open problem is the existence of such infinity of v's for $k = 13$. Here are examples of other interesting properties of PMD given in [9].

Proposition 4.

a) M is a PMD, $k > v/2 \Rightarrow M$ is trivioid $S(k-1,k,v)$.

But $k = v/2$ in cases $M = S(3,4,8)$, $M = S(5,6,12)$.

b) Let M be PMD, M* dual PMD and let $c(M)$ denote the minimum circuit cardinality in M. Then $c(M^*) \geq \max(r,c(M))$.

But $c(M^*) = r = c(M) = v/2$ for affine geometry $EG(3,2)$ and for Witt design $S(5,6,12)$.

Matroid M is self-dual if the hyperplanes of M* are identical with the hyperplanes of M (so, in particular, $r = v/2$).

Proposition 5 ([10]). PMD (which is not a trivial) is self-dual if and only if it is $S(k-1,k,2k)$ for some $k+1$ $(k>2)$ odd prime.

Only known examples of self-dual PMD (which is not a trivioid) are $S(1,2,4)$, $S(3,7,8)$ and $S(5,6,12)$.

From now on we will consider some extremal properties of sets of flats of PMD. Let us denote by $A(L,k,v)$ any system $B = \{A_j\}$ of k-subsets of given v-set V such that $|A_f \cap A_g| \in L$ for any different $A_f, A_g \in B$. We will call system $B = A(L,k,v)$ maximum if $|B| \geq |B'|$ for any other $B'=A(L,k,v)$. We will call system $B = A(L,k,v)$ maximal if $B \cup \{A\} = A(L,k,v) \Rightarrow A \in B$ for any k-subset A of V. Of course, any maximum A(L,k,v) is a maximal A(L,k,v).

It is evident that hyperplane family M^{r-1} of PMD is an A(L,k,v) with $|A(L,k,v)| = |M^{r-1}| = t(0,r-1,r)$.

Let us consider B = A(L,k,v). For the following propositions 6 - 9 we suppose $v \geq v_0(k)$.

Proposition 6 ([12]). $|B| \leq \prod_{i=0}^{r-2} \frac{v-\alpha_i}{k-\alpha_i}$.

Proposition 7 ([13]). $|B| = \prod_{i=0}^{r-2} \frac{v-\alpha_i}{k-\alpha_i} \Rightarrow$ B is the hyperplane family of a PMD.

Proposition 8 ([2],[3]). $|B| \geq c\, v^{|L|-1}$ for some constant $c = c_0(k)$ implies $(\alpha_i - \alpha_{i-1}) | (\alpha_{i+1} - \alpha_i)$ for $2 \leq i \leq r-1$.

It is an extension (for the case $v \geq v_0(k)$) of the divisibility property (2) from Proposition 3 to the class of "large" A(L,k,v)'s.

Proposition 9. Any M^i $(0<i<r)$ is complete, i.e. for any j, $0 \leq j \leq i$, there exist i-flats $F_1^i, F_2^i \in M^i$ such that $F_1^i \cap F_2^i$ is a j-flat.

From now on we remove the restriction $v \geq v_0(k)$. Thus the number of incomplete PMD's with given L,k is finite. For PMD S(t,k,v) all of them are known (Noda, Gross) and number of incomplete S(t,k,v) is finite with respect of t. M^i is complete in PG iff $2i \leq r$ and in AG iff $2i+1 \leq r$.

Also from Proposition 6 it follows that there exists only a finite number of PMD's with given L,k for which the hyperplane family is not a maximum A(L,k,v). I conjecture that in all these cases the hyperplane family is a maximal A(L,k,v). For all known PMD this is true. In fact trivioid and

Steiner system are maximum; for AG,PG and their any truncations it follows from [7] (as trmarked by B. Rothschild), for Hall-Young matroids it was remarked in [1] and also it is maximum (for $a \neq 4$) from results of [2], [3].

Now we consider another extremal property of PMD. Given a system C of flats (i.e., some subset of the lattice of all flats), we will call C a Sperner system if $A_1, A_2 \in C$, $A_1 \neq A_2 = A_1 \nsubseteq A_2$, $A_2 \nsubseteq A_1$. Sperner system C is maximum Sperner system if $|C| \geq |C'|$ for any other Sperner system. In Ch. 16.5 of [13] is given a result of Baker and following corollary of it.

Proposition 10. One of M^i $(0 \leq i \leq r)$ is maximum Sperner system of PMD.

In other words the lattice of all flats of PMD has the Sperner property. As immediate corollary we have

Proposition 10'. $|M^{r-1}|$ is the cardinality of maximum Sperner system of PMD with $r \geq v_0(k)$.

Cardinalities $|M^i|$ are denoted W_i $(0 \leq i \leq r)$ and called Whitney numbers (of second kind). The sequence $W_0, W_1, \ldots, W_2$ is called unimodal if $W_b \geq \min(W_a, W_c)$ for $a \leq b \leq c$.

Proposition 11 ([9]). The sequence of Whitney numbers of PMD is unimodal.

Moreover, Whitney numbers of PMD are log concave, i.e. $W_b \geq \sqrt{W_{b-1} W_{b+1}}$ for $2 \leq b \leq r-1$.

Some inequalities and characterisation theorems obtained for affine and projective geometries were recently generalized to PMD's. In [8] is given a characterisation of any set of i-flats of PMD (with given numbers of i-flats) as set of all i-flats contained in given flat for some special PMD (the necessary condition $\alpha_{i-1}^2 \le \alpha_i \alpha_{i-2}$ is to be compared with 3) of Proposition 3).

In [6] it was proved, in particular $\alpha_i \alpha_j \ge (\alpha_2 - 1)^{i-j}(\alpha_{i-1} - \alpha_{j-1})$ for $2 \le j < i \le r$, $r \ge 4$. The cases of equality and $r=4$ were studied in detail.

PMD-scheme will be any PMD such that M^{r-1} is subscheme of Johnson association scheme (two hyperplanes $F_1^{r-1}, F_2^{r-1} \in M^{r-1}$ are i-associated if $F_1^{r-1} \cap F_2^{r-1}$ is a (r-i)-flat. The examples of PMD-scheme are: any PMD of rank 3 (BIBD correspond to strongly regular graph); any tight $S(t,k,v)$ (moreover, any $S(t,k,v)$ with $\le t/2+1$ sizes of block intersection); $S(3,k,v)$-scheme studied by P. Cameron (he proved $v \le 2+k(k-1)(k-2)/2$; the known examples are Möbius planes $S(3,k,k^2-2k+2)$, $S(3,4,8)$, $S(3,6,22)$ and $S(3,12,112)$ if it exists); $AG(r,q)$, any truncation of $PG(r,q)$ being a q-analog of t-design.

On the other hand suppose given a $B=A(L,k,v)=\{A_i\}$ and let $\tilde{B}=\{a_i/\sqrt{k}\}$ where $a_i=(a_{i1},\ldots,a_{iv})$ is (0,1)-sequence representing A_i. So $\tilde{B}$ is a subset of unit sphere in R^v and $\{(a,b): a \ne b \in \tilde{B}\}=\{\alpha_i/k : \alpha_i \in L\}$. In terms of [14] $\tilde{B}$ is an A-set with $A=\{\alpha_i^2/k^2 : \alpha_i \in L\}$ and Propositions 6,7 mean that for $v \ge v_0(k)$:

a) $|B|=|\tilde{B}| \le (\frac{v}{k})^{\varepsilon} F(\frac{v}{k})$ (here $F(x)=x^{-\varepsilon} \prod_{\alpha \in A} \frac{x-\alpha}{1-\alpha}$ and $\varepsilon=1$ if $0 \in A$, $\varepsilon=0$ otherwise);

b) the equality holds iff B is M^{r-1} of some PMD of rank $r=|L|+1$. Note that $v/k=(a,a)$ for $a=k^{-1/2}(1,1,\ldots,1)$. In Th. 5.2,7.5 of [14] given (roughly speaking) an upper bound $f_{0\varepsilon}^{-1}$ for the cardinality of any A-set with equality only if it is association scheme ($f_{0\varepsilon}$ is the coefficient of $Q_{0\varepsilon}$ in the expansion $F(x) = \sum_{i=0}^{\infty} f_{i\varepsilon} Q_{i\varepsilon}(x)$ in the basic of the Jacobi polynomials $Q_{i\varepsilon}(x)$; $Q_{00}(x) = 1$, $Q_{01} = v$). From [15] follows that M^i in AG(r,q) is scheme iff i=1, r-1. From a result of N. Ito follows that S(4,5,v)-scheme is S(4,5,11).

Some open problems on PMD:

1) to find new exapmles of PND;

2) to prove the conjecture ([3]) that M^{r-1} of any PMD is maximal A(L,k,v);

3) there exists an infinity of v's such that there exists a PMD with $\alpha_0=0$, $\alpha_1=1$, $\alpha_2=3$, $\alpha_3=|F^{r-1}|=13$, $\alpha_4=|V|=v$?

4) to find the minimal i such that for i-truncated AG (or PG) M^{r-1} is maximum A(L,k,v). It has to be close to r/2.

5) for Hall-Young matroids: a) to find the minimal a such that for $v=3^a$ its M^3 is maximum; b) to describe the cases (to find the maximal a) such that for $v=3^a$ it is incomplete;

6) to describe PMD-schemes.

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